Math 259A Lecture 8 Notes

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1 von Neumann Bicommutant Theorem and Kaplansky's Density Theorem

1.1 von Neumann's bicommutant theorem

If $S \subseteq \mathcal{B}(H)$, we denote $S' = \{x \in \mathcal{B}(H) : xy = yx \, \forall y \in S\}$ to be the commutant of S. Last time, we had the following results.

Proposition 1.1. *S' is always strong-operator closed, S' is an algebra with unit, and if $S = S^*$, S' is a *-algebra.*

In particular, if $S = S^*$, then S' is a von Neumann algebra.

Theorem 1.1 (von Neumann's bicommutant theorem, 1929)**.** *Let $M \subseteq B(H)$ be a *-algebra with $1_M = \mathrm{id}_H$. Then M is a von Neumann algebra if and only if $M = (M')'$.*

Remark 1.1. Some people call this von Neumann's density theorem because it says that the weak operator closure (or equivalently, the strong operator closure) of M is $(M')'$.

Proposition 1.2. *If $A \subseteq B(H)$ is an algebra, then for all $\xi \in H$, $[A\xi]$ is invariant to all $a_0 \in A$.*

Corollary 1.1. *If $A = A^*$ is a *-algebra, then $[A\xi]$ is invariant to both a_0 and a_0^* for all $a_0 \in A$. Thus, $[A\xi]$ is reductive for a_0: $a_0 \cdot [A\xi_0] = [A\xi_0] \cdot a_0$. So $[A\xi] \in A'$.*

Now we can prove the theorem.

Proof. ($\Longrightarrow$): M'' is weak operator closed, so it is a von Neumann algebra.

($\Longleftarrow$): We have $M \subseteq M''$ and M'' is strong operator closed, so $\overline{M}^{\mathrm{so}} \subseteq M''$. We want to show that M is dense in M'': if $x'' \in M''$, then for any $\xi_1, \dots, \xi_m \in H$ and $\varepsilon > 0$, there is an $x \in M$ such that $\|(x - x'')\xi_i\| < \varepsilon$ for all i.

Step 1: Take first the case $n = 1$. Since $x'' \in M''$, $[x'', [M\xi]] = 0$ (where the outer bracket means commutator); that is, $x''[M\xi](\xi) = [M\xi]x''(\xi) \in \overline{M\xi}$. On the other hand, the left hand side is $x''(\xi)$. So $x''\xi \in \overline{M\xi}$. So for any $\varepsilon > 0$, there is an $x \in M$ such that $\|x''\xi - x\xi\| < \varepsilon$.

Step 2: For arbitrary n, take $\widetilde{M} \subseteq \mathcal{B}(H^{\oplus n})$, where $\widetilde{M}$ is the collection of block diagonal matrices that look like

$$\begin{bmatrix} x & & & \\ & x & & \\ & & \ddots & \\ & & & x \end{bmatrix}$$

for $x \in M$. Then $\widetilde{M}$ is SO closed in $\mathcal{B}(H^{\oplus n})$, and $\widetilde{M} \supseteq \{(x'_{i,j})_{1\leq i,j\leq n} : x'_{i,j} \in M'\}$ If $x = (y_{i,j})_{i,j} \in (\widetilde{M})'$ and

$$e_{i,i} = \begin{bmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix},$$

then $e_{i,i} \in \widetilde{M}'$. So $\widetilde{M}' = \{(x'_{i,j})_{1\leq i,j\leq n} : x'_{i,j} \in M'\}$. Now let

$$\widetilde{M''} = \left\{ \begin{bmatrix} x'' & & & \\ & x'' & & \\ & & \ddots & \\ & & & x'' \end{bmatrix} : x'' \in M'' \right\} \subseteq \{(x'_{i,j})_{1\leq i,j\leq n} : x'_{i,j} \in M'\}'.$$

Then $\widetilde{M''} \subseteq (\tilde{M}')' = \widetilde{M}''$. By the first part applied to

$$\widetilde{x''} = \begin{bmatrix} x'' & & & \\ & x'' & & \\ & & \ddots & \\ & & & x'' \end{bmatrix} \in \widetilde{M}'',$$

we have for $(\xi_1, \dots, \xi_n) \in H^n$ that $\widetilde{x''}\xi \in \widetilde{M}\widetilde{\xi}$. So for every $\varepsilon > 0$, there is an $x = \widetilde{x} \in M$ such that $\|(\widetilde{x''} - \widetilde{x})\widetilde{\xi}\| < \varepsilon$. So $(\sum \|(x'' - x)\xi_i\|^2)^{1/2} < \varepsilon$. $\square$

1.2 Kaplansky's density theorem

Theorem 1.2 (Kaplansky, late 50s)**.** *Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra, and let $M_0 \subseteq M$ be a SO-dense *-algebra. Then $(\overline{M_0}^{\mathrm{so}})_1 = (M)_1$. Moreso, $(\overline{M_{0,h}}^{\mathrm{so}})_1 = (M_h)_1$ and $(\overline{M_{0,+}}^{\mathrm{so}})_1 = (M_+)_1$.*

Proof. Step 1: First observe that $\overline{M_{0,h}}^{\mathrm{so}} = M_h$. Indeed, $\overline{M_{0,h}}^{\mathrm{wo}} = M_h$ since $x \mapsto x^*$ is WO-continuous. So if $x_i \to x = x^*$, then $\frac{x_i + x_i^*}{2} \to x$. So $\overline{M_{0,h}}^{\mathrm{so}} = \overline{M_{0,h}}^{\mathrm{wo}} = M_h$.

Step 2: Show that $(\overline{M_{0,h}}^{\mathrm{so}})_1 = (M_h)_1$. We can assume $M_0 = \overline{M_0}^{\mathrm{norm}}$. Let $x = x^* \in (M)_1$, so $\mathrm{Spec}(x) \subseteq [-1,1]$. Take the bijection $f : [-1,1] \to [-1,1]$ sending $t \mapsto \frac{2t}{1+t^2}$. Note that given any $b = b^*$, $f(b)$ makes sense and $\|f(b)\| \leq 1$. So there is a $y \in (M_h)_1$ such that $x = \frac{2y}{1+y^2}$. We have that there exist (by step 1) $y_i = y_i^* \xrightarrow{\mathrm{so}} y$ in $(M_0)_h$.

We claim that $\frac{2y_i}{1+y_i^2} \xrightarrow{\mathrm{so}} \frac{2y}{1+y^2} = x$. Indeed,

$$\begin{aligned}\left(\frac{2y_i}{1+y_i^2} - \frac{2y}{1+y^2}\right)\xi &= \frac{1}{1+y_i^2}\left(2y_i(1+y^2) - (1+y_i)^2 2y_i\right)\frac{1}{1+y^2}\xi \\ &= \frac{1}{1+y_i^2}\left((2y_i - 2y) + y_i(y-y_i)\right)\frac{2y}{1+y^2}\xi.\end{aligned}$$

The $2y_i - 2y$ part disappears because of the strong operator convergence, and the left term handles the rest. □

We will do the non self-adjoint part of the proof next time.